

Essentially Reductive Hilbert Modules II

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Abstract

Many Hilbert modules over the polynomial ring in m variables are essentially reductive, that is, have commutators which are compact. Arveson has raised the question of whether the closure of homogeneous ideals inherit this property and provided motivation to seek an affirmative answer. Positive results have been obtained by Arveson, Guo, Wang and the author. More recently, Guo and Wang extended the result to quasi-homogeneous ideals in two variables. Building on their techniques, in this note the author extends this result to Hilbert modules over certain Reinhardt domains such as ellipsoids in two variables and analyzes extending the result to the closure of quasi-homogeneous ideals in m variables when the zero variety has dimension one.

0 Introduction

In his study [1] of the m -shift Hilbert module H_m^2 over $\mathbb{C}[\mathbf{z}]$, the polynomial algebra in m variables, Arveson formulated a provocative conjecture which has attracted the attention of several researchers. He had established that the commutators of all operators and their adjoints defined on H_m^2 by module multiplication by polynomials in $\mathbb{C}[\mathbf{z}]$ belonged to the Schatten p -class \mathcal{L}^p for all $p > m$. He then conjectured that the same commutator condition held for all submodules in H_m^2 obtained as the closure $[I]$ in H_m^2 of a homogeneous ideal I and established the result in [2] in case I is generated by monomials.^{1,2} Using somewhat different methods, the author extended this latter result in [10] to a larger family of commuting weighted shifts in

¹The same result was proved earlier for the quotient defined by every homogeneous submodule in the Hardy space $H^2(\mathbb{D}^2)$ for the bidisk \mathbb{D}^2 by Curto, Muhly and Yan [9].

²Such Hilbert modules were defined to be essentially reductive in [12] and, later, essentially normal in [2].

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m -variables. At the same time, it was pointed out that these results suggested extending the conjecture to domains other than the unit ball \mathbb{B}^m in \mathbb{C}^m . The meaning of this latter statement becomes clear once one understands that H_m^2 can be realized as a reproducing Hilbert space of holomorphic functions on \mathbb{B}^m and that the commutator condition on H_m^2 can be shown to be equivalent to the same statement for the Bergman space $L_a^2(\mathbb{B}^m)$ on \mathbb{B}^m , that is, for the closure of $\mathbb{C}[\mathbf{z}]$ in the L^2 -space on \mathbb{B}^m relative to volume measure.

Subsequently, Guo obtained results in [15] which, when combined with earlier techniques of Arveson, established Arveson's original conjecture in H_2^2 for arbitrary homogeneous ideals in $\mathbb{C}[z_1, z_2]$. Then Guo, joined by Wang in [16], extended this result to general m when the complex dimension of the zero variety $Z(I)$ of I is one or, equivalently, when the Hilbert polynomial for the quotient module $H_m^2/[I]$ is linear (cf. [13]). Finally, in a recent paper [17] Guo and Wang showed that the closure of quasi-homogeneous ideals in a class of Hilbert modules, including H_2^2 , are essentially reductive.

In this note, based largely on the methods of Guo and Wang, we extend this latter result in two ways. First, we show that submodules obtained as the closure of quasi-homogeneous ideals in the Bergman space for certain Reinhardt domains in \mathbb{C}^2 are essentially reductive. Note that this result includes the case of the closure of homogeneous ideals, which is new. Second, we show that the same is true for the closure of certain, very restrictive quasi-homogeneous ideals in $\mathbb{C}[\mathbf{z}]$ for arbitrary m with one dimensional zero variety in the Bergman space for nice Reinhardt domains in \mathbb{C}^m . This collection of Reinhardt domains includes the ellipsoids of the form $E_{\mathbf{a}} = \left\{ \mathbf{z} \in \mathbb{C}^m : \sum_{i=1}^m a_i |z_i|^2 < 1 \right\}$ for \mathbf{a} in $[0, \infty)^m$. Of course, $E_{\mathbf{1}} = \mathbb{B}^m$ with $\mathbf{1} = (1, \dots, 1)$.

We will say that an ideal in \mathbb{C}^m is bivariate if it is generated by polynomials in two of the variables z_1, \dots, z_m at a time. Note that all ideals in $\mathbb{C}[z_1, z_2]$ are bivariate. The limit of the techniques in this note would seem to be bivariate ideals with one dimensional zero variety although we are unable to establish such a result at this time.

Most of the results in [2], [10], [15] and [16] apply not just to the closures of homogeneous ideals in a Hilbert space completion \mathcal{H} of $\mathbb{C}[\mathbf{z}]$ but to homogeneous submodules of $\mathcal{H} \otimes \mathbb{C}^k$. In this note, we confine our attention to the multiplicity one case, $k = 1$, or the closure of ideals.

We will assume the reader is familiar with [17] although we will provide statements of the relevant definitions, lemmas and propositions and the necessary proofs but emphasize mainly the points that are different and not straightforward extensions of those in [17].

1 The Basic Setup

A Reinhardt domain Ω in \mathbb{C}^m is one for which $\mathbf{z} = (z_1, \dots, z_m)$ in Ω implies $(e^{i\theta_1} z_1, \dots, e^{i\theta_m} z_m)$ is in Ω for all m -tuples $(e^{i\theta_1}, \dots, e^{i\theta_m})$ in the m -torus. Hence, the absolute values of the co-

ordinates of a point are sufficient to determine if the point is in Ω . Even so, Reinhardt domains can be somewhat pathological. A nice class of them is determined by smooth functions $\varphi: [0, \infty)^m \rightarrow [0, \infty)$ so that $\Omega_\varphi = \{\mathbf{z} \in \mathbb{C}^m: \varphi(|z_1|, \dots, |z_m|) < 1\}$. We assume further that φ is monotonically increasing in each variable t_i , $i = 1 \dots m$. In this case, the boundary $\partial\Omega_\varphi$ of Ω_φ is the set $\{\mathbf{z} \in \mathbb{C}^m: \varphi(|z_1|, \dots, |z_m|) = 1\}$. Moreover, Ω_φ is homeomorphic to \mathbb{B}^m and $\partial\Omega_\varphi$ to the unit sphere $\partial\mathbb{B}^m$. However, unlike the case of analogous domains in \mathbb{C} , there is, in general, no biholomorphism between one of them and the ball.

Now $\mathbb{C}[\mathbf{z}]$ is norm dense in the closure, $\mathcal{H}_\varphi = L_a^2(\Omega_\varphi)$, of the functions holomorphic on a neighborhood of the closure of Ω_φ for volume measure on Ω_φ . Moreover, since volume measure on Ω_φ restricts, in the sense of the Fubini Theorem, to Lebesgue measure on the natural m -torus through each point, we see that the monomials form an orthogonal basis for \mathcal{H}_φ . Further, Ω_φ is pseudo-convex if and only if it is logarithmically convex (cf. [18]). In addition, if $\partial\Omega_\varphi$ contains no disks (cf. [14]), then the commutators on \mathcal{H}_φ are in \mathcal{L}^p for $p > m$. (We will say that such Reinhardt domains are pseudo-convex without disks.) This implies that \mathcal{H}_φ satisfies the hypotheses of Theorem 4.3 in [10]. Hence, modules defined as the closure of ideals in $\mathbb{C}[\mathbf{z}]$ generated by monomials are p -essentially reductive for $p > m$. In this note, we show the same is true when the ideal is bivariate, quasi-homogeneous, has zero variety with complex dimension one and is *radical*. Note that in case if the ideal is actually homogeneous, then for some Reinhardt domains this result follows from the earlier work of Guo and Wang [16].

We assume in what follows that Ω_φ is a pseudo-convex Reinhardt domain without disks. A polynomial $p(\mathbf{z}) = \sum_{\mathbf{i}, \mathbf{j}=0}^{\infty} p_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \dots, i_m)$ is said to be quasi-homogeneous of degree ℓ with respect to the weights $\mathbf{n} = (n_1, \dots, n_m)$ if $p_{\mathbf{i}} \neq 0$ implies $\sum_{i=1}^m n_j i_j = \ell$.

If we consider the action of \mathbb{T}^m on Ω_φ in the obvious way, then quasi-homogeneity can be characterized in terms of this action. For each λ in \mathbb{R} , consider the action γ_λ on Ω_φ defined by

$$\gamma_\lambda(z_1, \dots, z_m) = (e^{in_1\lambda} z_1, \dots, e^{in_m\lambda} z_m).$$

Then the quasi-homogeneous polynomials of degree ℓ for the weights (\mathbf{n}) are just the eigenvectors for the action of $\{\gamma_\lambda\}_{\lambda \in \mathbb{R}}$ on \mathcal{H}_φ for the eigenvalue ℓ . Since the monomials form an orthogonal basis for \mathcal{H}_φ , we see for fixed weights (\mathbf{n}) that $\mathcal{H}_\varphi = \bigoplus \mathcal{H}_\varphi^\ell$, where the \mathcal{H}_φ^ℓ are the quasi-homogeneous polynomials of degree ℓ with respect to (\mathbf{n}) , and the decomposition is orthogonal. As indicated by Guo and Wang, if \mathcal{M} is the closure of a quasi-homogeneous ideal with weights (\mathbf{n}) , then

$$\mathcal{M} = \bigoplus \mathcal{M}_\ell \text{ and } \mathcal{M}^\perp = \bigoplus \mathcal{H}_\varphi^\ell \ominus \mathcal{M}_\ell,$$

where $\mathcal{M}_\ell = \mathcal{M} \cap \mathcal{H}_\varphi^\ell$ and $\mathcal{H}_\varphi^\ell \ominus \mathcal{M}_\ell = \mathcal{M}^\perp \cap \mathcal{H}_\varphi^\ell$. In other words, both \mathcal{M} and \mathcal{M}^\perp are graded.

Let \mathcal{M} be the closure in \mathcal{H}_φ of a quasi-homogeneous ideal I . For p in $\mathbb{C}[\mathbf{z}]$, let M_p denote the operator on \mathcal{H}_φ defined by module multiplication by p ; A_p the restriction of M_p to the

submodule \mathcal{M} , B_φ the action of M_p from \mathcal{M} to \mathcal{M}^\perp , and C_p the compression of M_p to \mathcal{M}^\perp , that is, $M_p = \begin{pmatrix} A_p & B_p \\ 0 & C_p \end{pmatrix}$.

Using the index formula for Toeplitz operators on \mathcal{H}_φ [9], one can show that there exists a Fredholm operator on $\mathcal{H}_\varphi \otimes \mathbb{C}^m$ with non-zero index. Hence, there exists a non-zero compact operator in the C^* -algebra, $C^*(\mathcal{H}_\varphi)$, generated by the operators defined by module multiplication on \mathcal{H}_φ . Further, since \mathcal{H}_φ is a reproducing kernel Hilbert space over Ω_φ which is connected, \mathcal{H}_φ is irreducible. Therefore, the ideal $\mathcal{K}(\mathcal{H}_\varphi)$ of all compact operators on \mathcal{H}_φ is contained in $C^*(\mathcal{H}_\varphi)$. Finally, the same index result implies that $C^*(\mathcal{H}_\varphi)/\mathcal{K}(\mathcal{H}_\varphi)$ is isomorphic to $C(\partial\Omega_\varphi)$ with the homomorphism extending the obvious map $M_p \rightarrow p|_{\partial\Omega_\varphi}$ for p in $\mathbb{C}[\mathbf{z}]$. (This is a known result but perhaps not quite in this generality [8].) Thus we have the short exact sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{H}_\varphi) \rightarrow C^*(\mathcal{H}_\varphi) \xrightarrow{\pi} C(\partial\Omega_\varphi) \longrightarrow 0.$$

where π is the quotient map.

2 Main Results

We recall a lemma from [2] and [10] which is one starting point for Guo and Wang [17]. For \mathcal{M} a subspace of the Hilbert space \mathcal{H} , let $P_{\mathcal{M}}$ denote the projection onto \mathcal{M} .

Lemma 2.1. *If \mathcal{H} is an essentially reductive Hilbert module over $\mathbb{C}[\mathbf{z}]$ and \mathcal{M} is a submodule of \mathcal{H} , then \mathcal{M} is essentially reductive iff \mathcal{H}/\mathcal{M} is essentially reductive iff the operators $P_{\mathcal{M}}M_pP_{\mathcal{M}^\perp}$ are compact for all p in $\mathbb{C}[\mathbf{z}]$.*

Next we recall another result from [10] which we use to establish an extension of Lemma 2.5 of [17]. The result in [10] is the dual of this one obtained by taking adjoints.

Lemma 2.2. *If $\mathcal{M}_0, \mathcal{M}_1$ and \mathcal{M}_2 are Hilbert modules over $\mathbb{C}[\mathbf{z}]$ with \mathcal{M}_1 and \mathcal{M}_2 essentially reductive and X_0 and X_1 are module maps so that X_0 is isometric and*

$$0 \longrightarrow \mathcal{M}_0 \xrightarrow{X_0} \mathcal{M}_1 \xrightarrow{X_1} \mathcal{M}_2 \longrightarrow 0$$

is exact, then \mathcal{M}_0 is essentially reductive.

Lemma 2.3. *If I and J are ideals in $\mathbb{C}[\mathbf{z}]$ so that $Z(I) \cap Z(J) \cap \partial\Omega_\varphi = \emptyset$ and both $[I]$ and $[J]$ are essentially reductive, then $[I \cap J]$ is essentially reductive.*

Proof. We begin by first defining the isometric module map X_0 : $[I \cap J] \longrightarrow [I] \oplus [J]$ so that $X_0f = \frac{1}{\sqrt{2}}(f \oplus -f)$. Next we define X_1 : $[I] \oplus [J] \longrightarrow [I, J]$ so that $X_1(f \oplus g) = f + g$ and observe that $0 \longrightarrow [I \cap J] \xrightarrow{X_0} \begin{smallmatrix} [I] \\ \oplus \\ [J] \end{smallmatrix} \longrightarrow [I, J] \longrightarrow 0$ is seen to be exact once we know that X_1 is

onto. To show that, let $\{p_i\}_{i=1}^s$ and $\{q_j\}_{j=1}^t$ be finite sets of generators for I and J , respectively. If we consider the row operators R and S from $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ and $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ defined by

$$R(f_1 \oplus \cdots \oplus f_s) = \sum_{i=1}^s p_i f_i \quad \text{and} \quad S(g_1 \oplus \cdots \oplus g_t) = \sum_{j=1}^t q_j g_j,$$

respectively, then $RR^* + SS^* = \sum_{i=1}^s M_{p_i} M_{p_i}^* + \sum_{j=1}^t M_{q_j} M_{q_j}^*$. Moreover, since $Z(I) \cap Z(J) \cap \partial\Omega_\varphi = \emptyset$, we have $\pi(RR^* + SS^*) = \sum_{i=1}^s |p_i|^2 + \sum_{j=1}^t |q_j|^2 > 0$ on $\partial\Omega_\varphi$ implying that $RR^* + SS^*$ is Fredholm. But the range of X_1 contains the span of the ranges of R and S , which has finite codimension in \mathcal{H}_φ . Thus, X_1 is onto the essentially reductive model $[I, J]$. \square

Note that one can modify the proof in [17] to establish that $[I \cdot J]$ is essentially reductive under the assumptions of this lemma but we won't need that result in this paper.

We now follow [17] in reducing the case of quasi-homogeneous submodules to principal ideals generated by a power of a prime. We begin with the case of two variables.

If \mathcal{M} is a quasi-homogeneous submodule of \mathcal{H}_φ , then $I = \mathcal{M} \cap \mathbb{C}[z_1, z_2]$ is a quasi-homogeneous ideal with $[I] = \mathcal{M}$. (The argument to establish this result depends on decomposing elements of \mathcal{M} into components by degree and then establishing that the components are also in \mathcal{M} .) The remainder of the proof rests on a result of Yang [19] which states that I has a Beurling form, $I = pL$, where p is the GCD of the polynomials in I and L is an ideal with finite codimension in $\mathbb{C}[z, w]$. Since L is quasi-homogeneous and $\mathbb{C}[z, w]/L$ is finite dimensional, one knows that $Z(L) = (0)$. Thus one can use Lemma 2.3 to conclude that $[p] \cap [L] = [pL]$ is essentially reductive if $[p]$ is.

Since a quasi-homogeneous polynomial $p(z_1, z_2)$ has the form $p(z_1, z_2) = \alpha z_1^r z_2^s \prod_i (z_1^{t_i} - \alpha_i z_2^{u_i})^{v_i}$ with the α_i non-zero and distinct [4], a repeated application of Lemma 2.3 reduces the essential reductivity of \mathcal{M} to submodules generated by ideals of the form $[z_1^r z_2^s]$ and $[(z_1^t - \alpha z_2^u)^v]$. For the first case, the result follows from the theorem for monomials in [10]. For the second, we need a further argument.

First, we extend Lemma 2.2 of Guo and Wang [17] to a Bergman space \mathcal{H}_φ over a pseudoconvex Reinhardt domain Ω_φ in \mathbb{C}^m without disks in its boundary. We divide the proof into two parts. For \mathcal{H} a Hilbert space, let π denote the quotient map from $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. (This notation is consistent with the earlier definition of π .)

Lemma 2.4. *If \mathcal{A}_I is the C^* -subalgebra of $C^*(\mathcal{H}_\varphi)$ generated by the operators M_p for p in an ideal I in $\mathbb{C}[\mathbf{z}]$ and $\mathcal{K}(\mathcal{H}_\varphi)$, then*

$$\mathcal{A}_I = \{T \in C^*(\mathcal{H}_\varphi) : \pi(T) \equiv 0 \text{ on } Z(I)\}.$$

Proof. Since $\pi(\mathcal{A}_I)$ is the self-adjoint subalgebra of $C(\partial\Omega_\varphi)$ generated by the restriction of the functions in I to $\partial\Omega_\varphi$, the result follows. \square

Lemma 2.5. Let I be an ideal in $\mathbb{C}[\mathbf{z}]$ so there exist positive constants $\mathbf{a} = (a_1, \dots, a_m)$ such that $M_{z_i}M_{z_i}^* - a_i$ is in \mathcal{A}_I for $i = 1, 2, \dots, m$. If \mathcal{M} is the closure of I in $\mathbb{C}[\mathbf{z}]$ and $Z(I) \cap \partial\Omega_\varphi \cap Z(z_1) = \emptyset$, then $C_{z_1}C_{z_1}^*$ is Fredholm and $C_{z_1}B_{z_1}^*$ is compact.

Proof. For f in I , we see that the matrix $M_f = \begin{pmatrix} A_f & B_f \\ 0 & 0 \end{pmatrix}$, since the range of M_f is contained in \mathcal{M} . But M_f is essentially normal from which it follows that B_f is compact for f in I .

By the Hilbert basis theorem, there exist polynomials q_1, \dots, q_s in I that generate I . Hence, we have $Z(I) = \{\mathbf{z} \in \mathbb{C}^m : q_1(\mathbf{z}) = \dots = q_s(\mathbf{z}) = 0\}$. Therefore, $|z_1|^2 + \sum_{i=1}^s |q_i(\mathbf{z})|^2 > 0$ on $\partial\Omega_\varphi$ and hence $T = M_{z_1}M_{z_1}^* + \sum_{i=1}^s M_{q_i}M_{q_i}^*$ is Fredholm.

Now $[T, P_{\mathcal{M}}]$ compact implies that $C_{z_1}C_{z_1}^*$ is Fredholm and $C_{z_1}B_{z_1}^*$ is compact using matrix calculations as in [17]. \square

Lemma 2.6. If, in addition to the hypotheses of Lemma 2.5, either $m = 2$ and I is the principal ideal generated by $(z_1^t - \alpha z_2^u)^v$ with $\alpha \neq 0$ or m is arbitrary and I is radical, then B_{z_1} is compact.

Proof. Since $C_{z_1}C_{z_1}^*$ is Fredholm, this will follow once one knows that C_{z_1} is Fredholm or that the null space of C_z is finite dimensional. We claim, in fact, that it's (0) in both cases.

If $C_{z_1}h = 0$ for some h in \mathcal{M}^\perp , then z_1h is in \mathcal{M} . We can decompose h in \mathcal{H}_φ so that $h = \Sigma h_\ell$ with h_ℓ in \mathcal{H}_φ^ℓ . But then $z_1h = \Sigma z_1h_\ell$ is in \mathcal{M} and since I is quasi-homogeneous, we see that z_1h_ℓ is in $\mathcal{M}_{\ell+1} \subset I \cap \mathcal{M} = I$. Hence, we have $Z(z_1) \cup Z(h_\ell) = Z(z_1h_\ell) \supset Z(I)$ and since $Z(z_1) \cap Z(I) = \emptyset$, it follows that $Z(I) \subset Z(h_\ell)$. Since I is radical, we have h_ℓ in \mathcal{M} , which implies $h_\ell = 0$ or $h = 0$ and the result is proved in this case.

If I is singly generated by $q(z_1, z_2) = (z_1^t - \alpha z_2^u)^v$, then we obtain $z_1h_\ell = qp$. Since z_1 and q are relatively prime, it follows that z_1 divides p and hence h_ℓ is in \mathcal{M}_ℓ and is 0. This is the argument in [17]. \square

We complete the proof of the main theorem in essentially the same way as in [17].

Theorem 1. If Ω_φ is a pseudo-convex Reinhardt domain in \mathbb{C}^2 without disks in its boundary, then every quasi-homogeneous submodule of \mathcal{H}_φ is essentially reductive.

Proof. The first requirement needed to apply Lemma 2.6 is to show somehow that $|z_1|$ is constant on $\partial\Omega_\varphi \cap Z(I)$. Since I is generated by $(z_1^t - \alpha z_2^u)^v$, we see that there exists a monotonically increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ so that (z_1, z_2) in $Z(I)$ implies $\psi(|z_1|) = |z_2|$. Thus if (z_1, z_2) is in $\partial\Omega_\varphi \cap Z(I)$ we have $\varphi(|z_1|, \psi(|z_1|)) = 1$. But this uniquely determines $|z_1|$ since φ is monotone in each variable. We can now apply Lemma 2.6 to conclude that B_{z_1} is compact. Reversing the roles of z_1 and z_2 yields that B_{z_2} is compact which completes the proof. \square

What kind of assumption can we make for $m > 2$ to establish the hypothesis of the constancy of the restriction of $|z_1|$ to $\partial\Omega_\varphi \cap Z(I)$? One possibility is the following definition.

An ideal I in $\mathbb{C}[\mathbf{z}]$ will be said to have an absolutely determining zero variety if for some fixed j_0 , $1 \leq j_0 \leq m$, and each i , $1 \leq i \leq m$, there is a continuous monotonically increasing function ψ_i so that $|z_i| = \psi_i(|z_{j_0}|)$ for each \mathbf{z} in $Z(I)$.

Note that one can show that if I is an ideal in $\mathbb{C}[z_1, z_2]$ generated by a quasi-homogeneous polynomial of the form $p(z_1, z_2) = (z_1^t - \alpha z_2^u)^v$ for nonnegative integers t, u and v , and α in $\mathbb{C} \setminus \{\mathbf{0}\}$, then I has an absolutely determining zero variety. Also, if I is an ideal in $\mathbb{C}[z_1, \dots, z_m]$ generated by such a polynomial q in z_1 and z_2 and the monomials z_3, z_4, \dots, z_m , then the same thing is true. Further, if $q_1(z_1, z_m)$ and $q_2(z_2, z_m), \dots, q_k(z_k, z_m)$ for $1 \leq k < m-1$ are each quasi-homogeneous polynomials of the same form and I is the ideal generated by them and the monomials z_{k+1}, \dots, z_{m-1} , then again I has an absolutely determining zero variety. Finally, other combinations are possible such as an ideal generated by quasi-homogeneous polynomials of the above form $p_1(z_1, z_2)$, $p_2(z_2, z_3)$ and $p_3(z_3, z_4)$ in $\mathbb{C}[z_1, \dots, z_4]$.

Continuing this line of thought one can show for I a bivariate, quasi-homogeneous, radical ideal having $\dim Z(I) = 1$, that I has an absolutely determining zero variety. Note that the fact that I is radical forces the generating polynomials to be prime factors having the form $z_i^t - \alpha z_j^u$ for $\alpha \neq 0$. The fact that $\dim Z(I) = 1$ forces the absolute values to all be related to that of one of the coordinates z_i . Let us provide some more details.

Let I be a bivariate ideal and set $I_i = I \cap \mathbb{C}[z_i]$ for $1 \leq i \leq m$ and $I_{i,j} = I \cap \mathbb{C}[z_i, z_j]$ for $1 \leq i, j \leq m$, $i \neq j$. (Here, we are viewing $\mathbb{C}[z_i]$ and $\mathbb{C}[z_i, z_j]$ as subalgebras of $\mathbb{C}[\mathbf{z}]$ in the obvious way.) Since $I_i \subset I_{i,j}$ for all $1 \leq j \leq m$, $j \neq i$, I is generated by the collection $[I_{i,j}]$, and we have $Z(I) = \cap Z(I_{i,j})$. Moreover, if I is a radical ideal, then so is each I_i and $I_{i,j}$. Finally, if I is quasi-homogeneous for the weights (\mathbf{n}) , then $I_{i,j}$ is quasi-homogeneous for the weights (n_i, n_j) .

If $I_i \neq (0)$, then it is generated by z_i and hence z_i is in I . If z_i is in I and $I_{i,j} \neq I_i$, then $I_j \neq (0)$ and $I_{i,j}$ is generated by z_i and z_j . Moreover, both z_i and z_j are in I . Partition the integers $\{1, 2, \dots, m\}$ into two sets Γ_1 and Γ_2 so that i is in Γ_1 if and only if z_i is in I . Then $\mathbb{C}[z_i : i \in \Gamma_1] \subset I$ and $Z(I) \subset \{\mathbf{z} \in \mathbb{C}^m : z_i = 0 \text{ for } i \in \Gamma_1\}$. We seek now to partition Γ_2 so that i and j in Γ_2 are equivalent if $I_{i,j} \neq (0)$. Since the ideal $I_{i,j} \subset \mathbb{C}[z_i, z_j]$ is generated by a prime polynomial $z_i^s - \alpha z_j^t$ for positive integers s and t and $\alpha \neq 0$, we see that we obtain an equivalence relation which partitions Γ_2 into subsets $\{\Gamma_2^k\}$ of $\{1, \dots, m\}$. For each Γ_2^k consider the ideal I'_k obtained from the intersection of I with $\mathbb{C}[z_i : i \in \Gamma_2^k]$. Again, I is generated by $\mathbb{C}[z_i : i \in \Gamma_1]$ and the collection $\bigcup_k I'_k$ and thus the zero variety $Z(I)$ is the intersection of the zero variety of $\mathbb{C}[z_i : i \in \Gamma_1]$ and the collection $\{Z(I'_k)\}$. Since $I'_k \neq \mathbb{C}[z_i : i \in \Gamma_2^k]$, we see that the dimension of $Z(I'_k)$ is strictly less than the cardinality of Γ_2^k . Since the dimension of $Z(I)$ is one, we see that there can only be one element in the partition of Γ_2 . Hence we can choose an i_0 in Γ_2 and define continuous, monotonically increasing functions $\{\psi_i\}_{i=1}^m$ from $[0, \infty)$ to $[0, \infty)$

so that for \mathbf{z} in $Z(I)$ we have $|z_i| = \psi_i(|z_{i_0}|)$ for $1 \leq i \leq m$. Note that $\psi_i \equiv 0$ for i in Γ_1 and $\psi_{i_0}(x) = x$.

It appears likely that a necessary condition for an ideal I to have an absolutely determining zero variety $Z(I)$, is for $Z(I)$ to have dimension one.

Now one can extend the argument from Theorem 1 to establish the following result. Note that when I is homogeneous and Ω_φ is symmetric in all the variables, a better result follows from [16].

Theorem 2. *If \mathcal{M} is the closure of a radical, bivariate, quasi-homogeneous ideal I in $\mathbb{C}[\mathbf{z}]$ with zero variety $Z(I)$ of dimension one, then \mathcal{M} is essentially reductive.*

Proof. Observe that if we consider a linear polynomial $p_{\mathbf{a}}(\mathbf{z}) = a_1 z_1 + \cdots + a_m z_m$ for \mathbf{a} in $\mathbb{C}^m \setminus \{\mathbf{0}\}$ satisfying $Z(I) \cap \partial\Omega_p \cap Z(p_{\mathbf{a}}) = \emptyset$, then it will follow from the preceding argument and Lemma 2.6 that $B_{p_{\mathbf{a}}}$ is compact. Since the dimension of $Z(I)$ is 1 and that of $Z(p_{\mathbf{a}})$ is $m - 1$, then the intersection $Z(I) \cap Z(p_{\mathbf{a}})$ consists of just the origin $(0, \dots, 0)$ for a dense open set U of \mathbf{a} in \mathbb{C}^m . This implies that $B_{p_{\mathbf{a}}}$ is compact for \mathbf{a} in U from which the result follows. \square

Can we eliminate the hypothesis that I is radical? Just as in the case of $\mathbb{C}[z_1, z_2]$, there are two steps: (1) handle the case in which the I_i are generated by a power of z_i and the $I_{i,j}$ by a power of the prime polynomial $(z_i^s - \alpha z_j^t)^v$ and (2) handle the case in which one allows $I_{i,j}$ to be generated by a product of powers of prime polynomials $\prod_k (z_i^{s_k} - \alpha_k z_j^{t_k})^{v_k}$.

One approach to the first step would be to relate the essential reductivity of a quasi-homogeneous bivariate ideal I with one dimensional zero variety $Z(I)$ to the essential reductivity of its radical \sqrt{I} . But we have made no progress in doing that even under the assumption that each $I_{i,j}$ is generated by a single prime polynomial. For the second step, one could use Lemma 2.3 to reduce to case 1 if one knew under what circumstances the ideal $I \cap J$ is generated by the collection $\{I_{i,j} \cap J_{i,j}\}$.

It is possible that the proof in [17] showing that a quasi-homogeneous submodule in H_2^2 is p -essentially reductive for $p > 2$, also carries over in the generality of Theorem 1 but the author has not verified that. It does seem likely that the result on identifying the K -homology class in $K_1(\partial\Omega_\varphi \cap Z(\mathcal{M}))$ carries over. One may need an extension of the index theorem for Toeplitz operators on strongly pseudo-convex domains due to Boutet de Monvel [5] to this more general class of Reinhardt domains.

Finally, it seems likely that this K -homology class agrees with the fundamental class defined by $\partial\Omega_\varphi \cap Z(\mathcal{M})$ as conjectured in [11].

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